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Simple Pairs of Parallel W-Surfaces.

BY HENRY DALLAS THOMPSON.

Formulas between the Elements of Parallel Surfaces.

Let x, y, z be the coordinates of any point M on a given surface S ; and let $X, Y, Z; E, F, G; D, D', D''; e, f, g; H, K; r_1, r_2$, etc., have the usual meaning on this surface; r_1, r_2 being measured *toward* the point M . [B.-L., p. 98]. (For convenience use will be made of the formulas of Bianchi, "Differential-geometrie," Deutsche Uebersetzung von Lukat, Leipzig, 1899, and such references will be indicated by B.-L.). Let l be a *constant* which gives the length (positive or negative) from some point M_0 on the normal at the point M ; then, if x', y', z' be the running coordinates,

$$(x' - x)X + (y' - y)Y + (z' - z)Z + l = 0 \quad (1)$$

is the equation to the plane parallel to the tangent plane at M , and at the distance $-l$ from M . To find the envelope S' of the plane (1), that is, the "parallel surface" to S with the constant distance l between the parallel tangent planes, differentiate (1) with regard to u and v respectively, which gives

$$\begin{aligned} (x' - x) \frac{\partial X}{\partial u} + (y' - y) \frac{\partial Y}{\partial u} + (z' - z) \frac{\partial Z}{\partial u} &= 0, \\ (x' - x) \frac{\partial X}{\partial v} + (y' - y) \frac{\partial Y}{\partial v} + (z' - z) \frac{\partial Z}{\partial v} &= 0. \end{aligned} \quad (2)$$

The solution of the equations (1) and (2) for $(x' - x), (y' - y), (z' - z)$ is (using B.-L., p. 128, mid.),

$$x' - x = -lX, \quad y' - y = -lY, \quad z' - z = -lZ. \quad (3)$$

Thus the point M' is the point M_0 , and the parallel surface S' is the dilated surface S_0 , of which Bonnet speaks in N. Annal. (1853), pp. 433–8.

Let x_0, y_0, z_0 ; X, Y, Z ; E_0, F_0, G_0 , etc., be the elements of the point M_0 of the surface S_0 corresponding to M of the surface S . Differentiation gives

$$\begin{aligned} \frac{\partial x_0}{\partial u} &= \frac{\partial x}{\partial u} - l \frac{\partial X}{\partial u}, & \frac{\partial y_0}{\partial u} &= \frac{\partial y}{\partial u} - l \frac{\partial Y}{\partial u}, & \frac{\partial z_0}{\partial u} &= \frac{\partial z}{\partial u} - l \frac{\partial Z}{\partial u}, \\ \frac{\partial x_0}{\partial v} &= \frac{\partial x}{\partial v} - l \frac{\partial X}{\partial v}, & \frac{\partial y_0}{\partial v} &= \frac{\partial y}{\partial v} - l \frac{\partial Y}{\partial v}, & \frac{\partial z_0}{\partial v} &= \frac{\partial z}{\partial v} - l \frac{\partial Z}{\partial v}. \end{aligned} \quad (4)$$

and thence follow directly from the definitions,

$$\begin{aligned} E_0 &= E + 2lD + l^2e, & F_0 &= F + 2lD' + l^2f, & G_0 &= G + 2lD'' + l^2g, \\ D_0 &= D + le, & D'_0 &= D' + lf, & D''_0 &= D'' + lg. \end{aligned} \quad (5)$$

Setting the values of e, f, g from

$$e + KE + HD = 0, \quad f + KF + HD' = 0, \quad g + KG + HD'' = 0 \quad [\text{B.-L., p. 119}]$$

in (5), these formulas become

$$\begin{aligned} E_0 &= E + 2lD - l^2(EK + DH), & D_0 &= D - l(EK + DH), \\ F_0 &= F + 2lD' - l^2(FK + D'H), & D'_0 &= D' - l(FK + D'H), \\ G_0 &= G + 2lD'' - l^2(GK + D''H), & D''_0 &= D'' - l(GK + D''H). \end{aligned} \quad (5')$$

Using the values of H and K [B.-L., p. 105 (18)]:

$$K = \frac{DD' - D'^2}{EG - F^2}, \quad H = \frac{2FD' - ED'' - GD}{EG - F^2},$$

direct calculation from the formulas (5') gives:

$$\begin{aligned} E_0G_0 - F_0^2 &= (EG - F^2) + 2l(ED'' + GD - 2FD') \\ &\quad + l^2[4(DD'' - D'^2) - K(EG + GE - 2F^2) - H(ED'' + GD - 2FD')] \\ &\quad - 2l^3[K(GD + ED'' - 2FD') + 2H(DD'' - D'^2)] \\ &\quad + l^4[K^2(EG - F^2) + HK(ED'' + GD - 2FD') + H^2(DD'' - D'^2)] \\ &= (EG - F^2)[1 - 2lH + l^2(4K - 2K + H^2) + 2l^3(KH - 2HK) \\ &\quad + l^4(K^2 - H^2K + H^2K)] \\ &= (EG - F^2)(1 - lH + l^2K)^2, \end{aligned}$$

$$\begin{aligned}
 D_0 D_0'' - D_0'^2 &= (DD'' - D'^2) - l[K(ED'' + GD - 2FD') + 2H(DD'' - D'^2)] \\
 &\quad + l^2[K^2(EG - F^2) + HK(ED'' + GD - 2FD') + H^2(DD'' - D'^2)] \\
 &= (EG - F^2)[K + l(KH - 2HK) + l^2(K^2 - H^2K + H^2K)] \\
 &= (EG - F^2)(1 - lH + l^2K)K, \\
 2F_0 D_0' - E_0 D_0'' - G_0 D_0 &= (2FD' - ED'' - GD) \\
 &\quad - l[-4(D'^2 - DD'') + 2K(F^2 - EG) + H(2FD' - ED'' - GD)] \\
 &\quad - l^2[K(6FD' - 3ED'' - 3GD) + H(6D'^2 - 3DD'' - 3D''D)] \\
 &\quad + 2l^3[K^2(F^2 - EG) + HK(2FD' - ED'' - GD) + H^2(D'^2 - DD'')] \\
 &= (EG - F^2)[H - l(4K - 2K + H^2) - l^2(3KH - 6HK) \\
 &\quad + 2l^3(-K^3 + H^2K - H^2K)] \\
 &= (EG - F^2)(1 - lH + l^2K)(H - 2lK).
 \end{aligned}$$

Combinations of these three formulas give:

$$K_0 = \frac{K}{1 - lH + l^2K}, \quad H_0 = \frac{H - 2lK}{1 - lH + l^2K}; \quad (6)$$

whence,

$$1 + lH_0 + l^2K_0 = \frac{1}{1 - lH + l^2K},$$

and, therefore, finally also,

$$K = \frac{K_0}{1 + lH_0 + l^2K_0}, \quad H = \frac{H_0 + 2lK_0}{1 + lH_0 + l^2K_0}. \quad (7)$$

The value of K_0 in (6) has been given by Craig, Jour. für Math., 94 (1883), pp. 162-170.

To the lines of curvature on S correspond the lines of curvature on S_0 ; for, from (5), if $F = 0$, $D' = 0$ and $f = 0$, then $F_0 = 0$ and $D'_0 = 0$. And the corresponding lines of curvature on the two surfaces have parallel tangents, for the direction cosines are in proportion, viz. [B.-L., p. 63]:

$$\cos(u, x) = \frac{1}{\sqrt{G}} \frac{\partial x}{\partial v}, \quad \cos(u, y) = \frac{1}{\sqrt{G}} \frac{\partial y}{\partial v}, \quad \cos(u, z) = \frac{1}{\sqrt{G}} \frac{\partial z}{\partial v}, \quad (8)$$

$$\cos(u, x_0) = \frac{1}{\sqrt{G_0}} \frac{\partial x_0}{\partial v}, \quad \cos(u, y_0) = \frac{1}{\sqrt{G_0}} \frac{\partial y_0}{\partial v}, \quad \cos(u, z_0) = \frac{1}{\sqrt{G_0}} \frac{\partial z_0}{\partial v}, \quad (9)$$

and setting in the values from (4) and using the formulas B.-L., p. 102 (13), the

cosines of (9) become :

$$\frac{1}{\sqrt{G_0}} \frac{\partial x}{\partial v} \left(1 - \frac{l}{r_1}\right), \quad \frac{1}{\sqrt{G_0}} \frac{\partial y}{\partial v} \left(1 - \frac{l}{r_1}\right), \quad \frac{1}{\sqrt{G_0}} \frac{\partial z}{\partial v} \left(1 - \frac{l}{r_1}\right),$$

and are proportional to the cosines of (8). And similarly for the other line of curvature.

Four Triples of Parallel W-Surfaces.

The equations (6) lead to pairs of simple parallel *W*-surfaces when $K_0 = \pm K$ or $H_0 = \pm H$.

Case a. If $K_0 = K$, the first equation of (6) gives $l = H/K = r_1 + r_2$. This equation, $l = r_1 + r_2$, may be called the Weingarten equation for this surface. In this case, $H_0 = -H$. (The solution $l = 0$ corresponds to the original surface.) Then $r_1 = l - r_2 = -r_{02}$, and $r_2 = l - r_1 = -r_{01}$. The surface S' at the distance $-\frac{1}{2}l$ from S , that is, the surface midway between S and S_0 will have, from (6), the mean curvature $H' = 0$; and then from (7),

$$H = \frac{lK'}{1 + \frac{1}{4}l^2K'}, \quad K = \frac{K'}{1 + \frac{1}{4}l^2K'}.$$

This may be expressed: The two parallel surfaces on the two sides of a minimal surface at any constant distances $\pm \frac{1}{2}l$, have at corresponding points their principal curvatures equal, and the mean curvatures equal but with opposite signs, and each radius of principal curvature equal to the negative of the non-corresponding radius of principal curvature of the other surface. A figure showing these relations may be obtained by taking two equidistant surfaces parallel to the catenoid.

Case b. If $K_0 = -K$, the first of (6) gives $K = 0$, or

$$l^2K - lH + 2 = 0. \tag{10}$$

Then the original *W*-surface being given by the Weingarten equation (10), the mid-surface S' at the distance $-\frac{1}{2}l$ will have, from (6) and (10), $K' = -4/l^2$. And, from (7), the two parallel surfaces at the constant distances $\pm \frac{1}{2}l$ from a surface with constant principal curvature $-4/l^2$, at corresponding points, have their principal curvatures equal but of opposite signs. A figure showing these

relations may be obtained by taking surfaces parallel to a surface of revolution with constant negative curvature.

Case c. If $H_0 = H$, the second of (6) gives the Weingarten equation

$$l = H/K - 2/H = (r_1^2 + r_2^2)/(r_1 + r_2), \quad (11)$$

and for the second surface, $-l = (r_{02}^2 + r_{01}^2)/(r_{02} + r_{01})$. To get the mid-surface S' , the combination of the two formulas (7) gives: $H'/K' = H/K - l$. And (11) then becomes, using the second of (7),

$$l = H'/K' + l - 2(1 + \frac{1}{2}lH' + \frac{1}{2}l^2K')/(H' + lK'),$$

$$\text{or} \quad H'^2 + lH'K' = 2K' + lH'K' + \frac{1}{2}l^2K'^2,$$

$$\text{or finally} \quad \frac{1}{2}l^2 = (H'^2 - 2K')/K'^2 = r_1'^2 + r_2'^2.$$

Case d. If $H_0 = -H$, the second of (6) gives $(lK - H)(lH - 2) = 0$, and the first factor equal to zero has been considered under case *a*. The consideration of the second factor gives the Weingarten equation $l = 2/H = 2r_1r_2/(r_1 + r_2)$. This surface, H equal to a constant, has been studied by Bonnet,* who has shown, as follows also at once from (6), that, for the mid-surface S' , $K' = 4/l^2 = H^2$. The two cases *b* and *d* give one pair of *real* surfaces parallel to and equidistant from *every* real surface with constant measure of curvature. Several well known theorems, some in an extended form, follow from (5) and the formulas immediately following. Every pair of orthogonal lines on one surface will correspond to a pair of orthogonal lines on a parallel surface only when the surfaces have constant mean curvature equal but with opposite signs. For, if $F_0 = 0$ and $F = 0$, with D' not zero and f not zero, then $2D' + lf = 0$ and $0 = f + HD'$; hence, $l = 2/H$. The asymptotic lines on one surface correspond to a pair of conjugate lines on the parallel surface only when the surfaces have constant mean curvature, equal but with opposite signs. For, from (5),

$$D_0D'' + D_0'D - 2D_0'D' = 2(DD'' - D'^2) + l(eD'' + gD - 2D'f).$$

* Bonnet, *Nouv. Ann.*, 12 (1853), pp. 433-8. See also: Jellett, *Jour. de Math.*, 18 (1853), pp. 163-7. Bour, *Éc. Polyt.*, 39 (1862), pp. 109 f. Bonnet, *Éc. Polyt.*, 42 (1867), p. 77. Simon, *Diss.*, Halle, 1876. Willgrad, *Diss.*, Göttingen, 1883. Chini, *Gior. di Mat.*, 27 (1889), pp. 107-123. Darboux, "Surfaces," vol. II, pp. 243 ff.; vol. III, pp. 375 f. Vivanti, *Lomb. Rend.* (2), 27 (1895), p. 699.

And the left-hand side of this equation equated to zero is the necessary and sufficient condition for the correspondence of the asymptotic with a pair of conjugate lines. [B.-L., p. 293.] But the right-hand side equated to zero gives $l = 2/H$. [B.-L., p. 124, (8)]. A pair of orthogonal lines on one surface corresponds to a pair of conjugate lines on a parallel surface only when the first surface has constant mean curvature and the second surface is the parallel mid-surface with constant principal curvature. For, $F = 0$ and $D'_0 = 0$ in (5) and the following formulas give: $0 = D' + lf$, $f + HD' = 0$ or $l = 1/H$.

Pairs of Anharmonic Parallel Surfaces.

The constant distance M_0M in the case d is $2/H$. If r_1 be M_1M and r_2 be M_2M , it is seen at once that here M_0 is the fourth harmonic to the points M_1, M, M_2 . The question at once arises, for what surface will the line M_0M be a constant, when M_0 is the fourth *anharmonic* to the points M_1, M, M_2 ? Here, a and b being constants,

$$r_1 : r_2 = aM_1M_0 : -bM_2M_0.$$

Setting in $M_1M_0 = r_1 - M_0M$, $M_2M_0 = r_2 - M_0M$, this reduces to $M_0M(ar_2 + br_1) = (a + b)r_1r_2$, or finally :

$$M_0M = \frac{a + b}{\frac{a}{r_1} + \frac{b}{r_2}}.$$

In order that M_0M shall be a constant, it is necessary and sufficient that the denominator be constant, say c ; that is, $M_0M = (a + b)/c$, and the Weingarten equation is the most general linear relation between the reciprocals of the principal radii of curvature,

$$\frac{a}{r_1} + \frac{b}{r_2} = c. \quad (12)$$

Let this surface be referred to its lines of curvature; the parallel surface at the distance $-M_0M$ or $-l$, will have from (5) and B.-L., p. 102, (14) and B.-L., p. 235, formulas between (3) and (4),

$$\begin{aligned} E_0 &= E(1 - l/r_2)^2, & G_0 &= G(1 - l/r_1)^2, \\ D_0 &= D(1 - l/r_2), & D_0'' &= D''(1 - l/r_1), \end{aligned}$$

and then $r_{01} = r_1(1 - l/r_1)$ and $r_{02} = r_2(1 - l/r_2)$, or

$$r_1 = r_{01} + l, \quad r_2 = r_{02} + l. \quad (13)$$

These values set in (12) give :

$$a(r_{02} + l) + b(r_{01} + l) = c(r_{01} + l)(r_{02} + l),$$

$$\text{or} \quad (a - cl)r_{02} + (b - cl)r_{01} - cr_{01}r_{02} = l(cl - a - b).$$

But setting in the value $l = (a + b)/c$ and dividing by $-r_{01}r_{02}$, this becomes the anharmonic surface, with the linear Weingarten equation :

$$\frac{b}{r_{01}} + \frac{a}{r_{02}} = -c. \quad (14)$$

The relation between the principal radii of curvature of all surfaces parallel to the original surface satisfying the equation (12) may be found. To this end, the values (13) set in (12) give again :

$$a(l + r_{02}) + b(l + r_{01}) = c(l + r_{01})(l + r_{02})$$

$$\text{or} \quad \frac{\{(a + b) - cl\}l}{r_{01}r_{02}} + \frac{a - cl}{r_{01}} + \frac{b - cl}{r_{02}} - c = 0, \quad (15)$$

From this equation it follows : There is one and only one other surface parallel to the surface satisfying (12) which has a linear relation between the reciprocals of the principal radii of curvature, and that is the surface at the distance l given by $(a + b) - cl = 0$, or $l = (a + b)/c$; and for this value of l , $a - cl = -b$, and $b - cl = -a$, so that (15) in this case becomes :

$$\frac{b}{r_{01}} + \frac{a}{r_{02}} + c = 0. \quad (14)$$

It is further seen from (15), that no parallel surface to (12) has a constant K_0 or a linear relation between H_0 and K_0 (unless $a = b$, where (12) reduces to case d above).

The surface midway between (12) and (14) is found by setting $l = (a + b)/2c$ in the formula (15), and this gives :

$$\frac{(a + b)^2}{4c} \cdot \frac{1}{r'_1 r'_2} + \frac{a - b}{2} \left(\frac{1}{r'_1} - \frac{1}{r'_2} \right) - c = 0. \quad (16)$$

And the pairs of parallel surfaces on each side of (16) at the distances $\pm l'$ respectively, satisfy the equations :

$$\frac{1}{c} \left\{ \frac{(a+b)^2}{4} - c^2 l'^2 \right\} \frac{1}{r_1 r_2} + \frac{a-b}{2} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \mp c l' \left(\frac{1}{r_1} + \frac{1}{r_2} \right) - c = 0. \quad (17)$$

It will have been noticed that the Weingarten equations for the *pairs* of parallel surfaces (not mid-surfaces) in cases a , b , c , and d , and also in the anharmonic case (17), can be derived one from the other by writing $-r_{01}$ and $-r_{02}$ for r_2 and r_1 respectively.

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